Combinatorial Burnside groups

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October 13, 2022

Outline

Introduction

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abelian groups receiving invariants of actions of arbitrary finite groups G, generalizing $\mathcal{B}_n(G)$. These record geometric information about subvarieties with nontrivial stabilizers, as well as characters of the action of stabilizers.

Introduction

- **2021:** Kresch and Tschinkel introduce a combinatorial version,

$$\mathcal{BC}_n(G),$$

forgetting the geometry of strata with nontrivial stabilizers.

$\mathcal{B}_n(H)$, with H abelian

Generated by symbols

$$\beta = (b_1, \ldots, b_n), \quad b_i \in H^\vee, \quad \langle b_1, \ldots, b_n \rangle = H^\vee,$$

with relations:

(O) $\beta = \beta^\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(n)})$;

(B) **for** $b_1 \neq b_2$**:** $\beta = \beta_1 + \beta_2$, where

$$\beta_1 = (b_1 - b_2, b_2, \ldots, b_n), \quad \beta_2 = (b_1, b_2 - b_1, \ldots, b_n),$$

for $b_1 = b_2$**:** $\beta = (b_2, b_2, \ldots, b_n) = (0, b_2, \ldots, b_n)$

$\mathcal{BC}_n(G)$, with G general

Generated by symbols

$$(H, Y, \beta),$$

where

- $H \subseteq G$ is an abelian group,
- $H \subseteq Y \subseteq Z_G(H)$, and
- $\beta = (b_1, \ldots, b_r)$, with $1 \leq r \leq n$,

subject to relations . . .

Relation (B2)

for $b_1 = b_2$**:** $(H, Y, (b_1, \dots, b_r)) = (H, Y, (b_2, \dots, b_r))$;
for $b_1 \neq b_2$**:**

$$(H, Y, \beta) =$$
$$\begin{cases} (H, Y, \beta_1) + (H, Y, \beta_2) & \text{if } b_i \in \langle b_1 - b_2 \rangle, \text{ for some } i, \\ \underbrace{(H, Y, \beta_1) + (H, Y, \beta_2)}_{\Theta_1} + \underbrace{(\bar{H}, Y, \bar{\beta})}_{\Theta_2} & \text{otherwise.} \end{cases}$$

$$\beta_1 := (b_1 - b_2, b_2, b_3, \dots, b_r), \quad \beta_2 := (b_1, b_2 - b_1, b_3, \dots, b_r),$$
$$\bar{H} := \ker(\langle b_1 - b_2 \rangle) \subseteq H, \quad \bar{\beta} := \beta|_{\bar{H}}.$$

Main Result

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- the sum is over G-conjugacy classes $[H, Y]$ of pairs (H, Y), with $H \subseteq G$ an abelian subgroup and $H \subseteq Y \subseteq Z_G(H)$.
- $$\mathcal{B}_n([H, Y]) \simeq \mathcal{B}_n(H)/(\mathbf{C}_{(H,Y)}),$$

  and $\mathbf{C}_{(H,Y)}$ is a certain conjugation relation.

Comparison with $\mathcal{B}_n(H)$ and $\mathcal{B}_n([H, Y])$

When G is abelian, then

$$\mathcal{B}_n([H,Y]) \simeq \mathcal{B}_n(H),$$

$$\mathcal{BC}_n(G) = \bigoplus_{H' \subseteq G} \bigoplus_{H'' \subseteq H'} \mathcal{B}_n(H'').$$

In general,

$$\mathcal{B}_n([H,Y]) \simeq \mathcal{B}_n(H)/(\mathbf{C}_{(H,Y)}),$$

where $(\mathbf{C}_{(H,Y)})$: for all β and $g \in N_G(H) \cap N_G(Y)$ we have

$$\beta = \beta^g.$$

Definition of $\mathcal{BC}'_n(G)$

In the proof, we introduce another group

$$\mathcal{BC}'_n(G)$$

generated by

$$(H, Y, \beta)',$$

with a modified relation (**B2′**):

for $b_1 = b_2$**:**

$$(H, Y, (b_1, b_2, \ldots, b_r))' = (H, Y, (b_2, \ldots, b_r))';$$

for $b_1 \neq b_2$**:**

$$(H, Y, \beta)' = (H, Y, \beta_1)' + (H, Y, \beta_2)'.$$

Isomorphism

Define $\mathbb{Z}$-linear maps on the respective groups:

$$\Psi : (H, Y, \beta) \mapsto \sum_{H' \subseteq H} (H', Y, \beta')',$$

$$\Phi : (H, Y, \beta)' \mapsto \sum_{H' \subseteq H} \mu(H', H)(H', Y, \beta'),$$

where

$$\beta' = \beta|_{H'},$$

and μ is the Moebius function on the lattice of subgroups of H.

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$$\mathcal{BC}_n(G) \simeq \mathcal{BC}'_n(G).$$

Computations

- Compute $\mathcal{BC}_n(G)$ for many G and $n \leq 5$.

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- Compute $\mathcal{BC}_n(G)$ for many G and $n \leq 5$.
- $G = \mathfrak{D}_p$, with $p \geq 5$ is a prime.

$$\mathcal{BC}_2(G) = \mathcal{B}_2([\mathfrak{C}_p, \mathfrak{C}_p]) \stackrel{?}{=} \mathbb{Z}^{\frac{(p-5)(p-7)}{24}} \times (\mathbb{Z}/2)^{\frac{p-3}{2}} \times \mathbb{Z}/\frac{p^2-1}{12}.$$

In fact,

$$\mathcal{B}_2^-(\mathfrak{C}_p) \otimes \mathbb{Q} \simeq \mathcal{B}_2([\mathfrak{C}_p, \mathfrak{C}_p]) \otimes \mathbb{Q},$$

with rank $g(X_1(p))$.

Computations of $\mathcal{BC}_*(G)$

- Nonabelian subgroups of the plane Cremona group.
- G with *primitive* actions on $\mathbb{P}^2$:

$$\mathfrak{A}_5, \mathrm{ASL}_2(\mathbb{F}_3), \mathrm{PSL}_2(\mathbb{F}_7), \mathfrak{A}_6.$$

- $G = \mathfrak{A}_5$,

$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^3 \quad \text{and} \quad \mathcal{BC}_n(G) = 0,\ n \geq 3.$$

- $G = \mathfrak{C}_3^2 : \mathrm{SL}_2(\mathbb{F}_3) = \mathrm{ASL}(2,3)$,

$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^7 \times \mathbb{Z}^{13},\ \mathcal{BC}_3(G) = \mathbb{Z}/2 \times \mathbb{Z},\ \mathcal{BC}_n(G) = 0, n \geq 4.$$

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- $G = \mathfrak{A}_6$,

$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^7 \times \mathbb{Z}/4 \times \mathbb{Z},\ \mathcal{BC}_3(G) = \mathbb{Z}/2 \times \mathbb{Z},\ \mathcal{BC}_n(G) = 0, n \geq 4.$$