Equivariant Birational Geometry of Linear Actions

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Introduction

The classification of actions of finite groups on algebraic varieties, up to equivariant birationality, has a long history. New invariants were introduced in [1], for abelian groups, and in [4] and [2], in general. These take values in abelian groups, defined by explicit generators and relations. The invariants record information about subvarieties with nontrivial stabilizers, as well as weights of the stabilizers in the normal bundles to the subvarieties. There are three types of groups,

$$\mathcal{B}_n(G), \quad \mathcal{BC}_n(G), \quad \mathrm{Burn_n(G)},$$

respectively. In the first case, G is assumed to be abelian, and only subvarieties with maximal stabilizer G are taken into account. In the second case, one records subvarieties with arbitrary abelian stabilizers and in addition the group acting on the subvariety. In the third case, one also records the equivariant birational type of that subvariety. The class of the G-action is computed on a *standard* model, where all stabilizers are abelian. The groups $\mathcal{B}_n(G)$ and $\mathcal{BC}_n(G)$ have finitely many generators and relations, and are effectively computable, which is generally not the case for $\mathrm{Burn_n(G)}$. All three groups have an interesting combinatorial structure. Our main results are:

- A decomposition theorem determining $\mathcal{BC}_n(G)$ in terms of $\mathcal{B}_n(H)$, with abelian $H \subseteq G$.
- An implementation in `magma` of the formalism of De Concini-Procesi compactifications of subspaces arrangements, adopted to equivariant setting to compute the class
$$[\mathbb{P}(V) \circlearrowleft G] \in \mathrm{Burn_n(G)},$$
of a faithful G-action on the projectivization of an $(n+1)$-dimensional G-representation V.
- Examples demonstrating the power and limitations of new invariants.

Definitions

The group $\mathcal{B}_n(H)$, with H **abelian**, is generated by sequences

$$\beta = (b_1, \dots, b_n), \quad b_i \in H^\vee, \quad \langle b_1, \dots, b_n \rangle = H^\vee,$$

of characters of H, up to order, with relation:

(B) blow-up:

for $b_1 \neq b_2$**:** $\beta = \beta_1 + \beta_2$, where

$$\beta_1 = (b_1 - b_2, b_2, \dots, b_n), \quad \beta_2 = (b_1, b_2 - b_1, \dots, b_n). \qquad (\star)$$

for $b_1 = b_2$**:** $\beta = (b_2, b_2, \dots, b_n) = (0, b_2, \dots, b_n)$.

The group $\mathcal{BC}_n(G)$, with G arbitrary, is generated by symbols

$$(H, Y, \beta),$$

where

- $H \subseteq G$ is abelian, $H \subseteq Y \subseteq Z_G(H)$, and
- $\beta = (b_1, \dots, b_r)$, a sequence of non-trivial characters of H, up to order,

with relations:

(C) conjugation: for all H, Y, and β, we have

$$(H, Y, \beta) = (gHg^{-1}, gYg^{-1}, \beta^g),$$

where β^g is the g-conjugate of β,

(V) vanishing:

$$(H, Y, \beta) = 0,$$

if $b_1 + b_2 = 0$, for some b_1, b_2 in β,

(B′) blowup:

for $b_1 \neq b_2$**:**

$$(H, Y, \beta) = \begin{cases} (H, Y, \beta_1) + (H, Y, \beta_2) & \text{if } b_i \in \langle b_1 - b_2 \rangle, \text{ for some } i, \\ (H, Y, \beta_1) + (H, Y, \beta_2) + (\bar{H}, Y, \bar{\beta}) & \text{otherwise.} \end{cases}$$

Here β_1, β_2 are as in $(\star)$ and

$$\bar{H} := \ker(\langle b_1 - b_2 \rangle) \subseteq H, \quad \bar{\beta} := \beta|_{\bar{H}}.$$

for $b_1 = b_2$**:**

$$(H, Y, (b_1, \dots, b_r)) = (H, Y, (b_2, \dots, b_r)).$$

The group $\mathrm{Burn}_n(G)$, with G arbitrary, is generated by symbols

$$(H, Y/H \circlearrowright K, \beta),$$

where

- H, Y, and β are as above, and
- K is a function field of an algebraic variety of dimension $n - r$, which carries a generically free action of Y (subject to some conditions, see [2, Section 4]).

The defining relations of $\mathrm{Burn}_n(G)$ are similar to those for $\mathcal{BC}_n(G)$.

Decomposition of $\mathcal{BC}_n(G)$

We define a conjugation relation: $(\mathbf{C}_{(H,Y)})$: $\beta = \beta^g$, for all $g \in N_G(H) \cap N_G(Y)$. By [6],

$$\mathcal{BC}_n(G) \simeq \bigoplus_{[H,Y]} \mathcal{B}_n(H)/(C_{(H,Y)}),$$

summing over G-conjugacy classes of pairs (H, Y), with $H \subseteq G$ abelian and $H \subseteq Y \subseteq Z_G(H)$. In particular, if G is abelian, then

$$\mathcal{BC}_n(G) = \bigoplus_{H' \subseteq G} \bigoplus_{H'' \subseteq H'} \mathcal{B}_n(H'').$$

Examples

Recall the groups admitting *primitive* actions on $\mathbb{P}^2$: $\mathfrak{A}_5, \mathrm{ASL}_2(\mathbb{F}_3), \mathrm{PSL}_2(\mathbb{F}_7), \mathfrak{A}_6$. We have:

- $G = \mathfrak{A}_5$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^3, \quad \mathcal{BC}_n(G) = 0,\ n \geq 3.$$
- $G = \mathrm{ASL}_2(\mathbb{F}_3)$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^7 \times \mathbb{Z}^{13},\ \mathcal{BC}_3(G) = \mathbb{Z}/2 \times \mathbb{Z},\ \mathcal{BC}_n(G) = 0, n \geq 4.$$
- $G = \mathrm{PSL}_2(\mathbb{F}_7)$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^3 \times \mathbb{Z}, \quad \mathcal{BC}_3(G) = \mathbb{Z}/2, \quad \mathcal{BC}_n(G) = 0, n \geq 4.$$
- $G = \mathfrak{A}_6$,
$$\mathcal{BC}_2(G) = (\mathbb{Z}/2)^7 \times \mathbb{Z}/4 \times \mathbb{Z},\ \mathcal{BC}_3(G) = \mathbb{Z}/2 \times \mathbb{Z},\ \mathcal{BC}_n(G) = 0, n \geq 4.$$

Applications

Using $\mathrm{Burn_n(G)}$ and $\mathcal{BC}_n(G)$, we give an example of nonbirational linear G-actions. Let G be the extension of C_3 by C_m^2. Consider its action on $\mathbb{P}^2$ given by

$$g_1 := (\zeta_m^{s_0} x_0 : x_1 : x_2), g_2 := (x_0 : \zeta_m^{s_1} x_1 : x_2), (x_2 : x_0 : x_1),$$

where ζ_m is a m-th root of unity and s_0, s_1 are positive integers coprime to m.

When $m = 5$, choosing $s_0 = 1, s_1 = 2$ and $s_0' = 3, s_1' = 4$, we obtain different linear G-representations V, V', with induced faithful actions on $\mathbb{P}^2 = \mathbb{P}(V), \mathbb{P}(V')$. There is a unique rank 2 abelian subgroup $T = C_m^2 \subset G$. The point $[0 : 0 : 1]$ is fixed under T.

The Reichstein-Youssin invariant, i.e., the determinant of the weights of T in the tangent space at the fixed point [5], gives

$$(s_0, 0) \wedge (0, s_1) = s_0 s_1 = s_0' s_1' = (s_0', 0) \wedge (0, s_1') \in \wedge^2(C_m^2) = C_m,$$

since $1 \cdot 2 = 3 \cdot 4 \mod 5$. Thus, the restriction to H does not allow to distinguish these G-actions. On the other hand, from [3, Theorem 8.4, 8.5], we have

$$\begin{aligned}[\mathbb{P}(V) \circlearrowleft G] =& (1, G \circlearrowright k(\mathbb{P}^2), ()) \\ &+ (C_5, C_5 \circlearrowright k(t), (2)) + (C_5, C_5 \circlearrowright k(t), (3)) \\ &+ (C_5^2, 1 \circlearrowright k, ((1,0),(0,3))) + (C_5^2, 1 \circlearrowright k, ((4,0),(0,2)))\end{aligned} \qquad (1)$$

$$\begin{aligned}[\mathbb{P}(V') \circlearrowleft G] =& (1, G \circlearrowright k(\mathbb{P}^2), ()) \\ &+ (C_5, C_5 \circlearrowright k(t), (2)) + (C_5, C_5 \circlearrowright k(t), (3)) \\ &+ (C_5^2, 1 \circlearrowright k, ((3,0),(0,1))) + (C_5^2, 1 \circlearrowright k, ((2,0),(0,4)))\end{aligned} \qquad (2)$$

where the group C_5^2 is generated by actions g_1, g_2 and the stabilizer group C_5 within symbols in $(1), (2)$ is generated by $g_1^{s_1} g_2^{s_0}$, fixing a projective line.

Considering the image of the difference under the natural homomorphism

$$\mathrm{Burn}_2(G) \to \mathcal{BC}_2(G) = (\mathbb{Z}/2)^2 \times (\mathbb{Z}/30)^2 \times \mathbb{Z}^{19},$$

we find that it equals

$$T_2^2 + 13T_{30}^2 + e_3 - e_7 + e_8 + 2e_{11} - e_{12} + e_{16} \neq 0 \in \mathcal{BC}_2(G),$$

where $T_2^{1,2}, T_{30}^{1,2}$ and $e_i, i = 1, \dots, 19$ are generators for $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/30)^2$, and the torsion-free part, respectively.

Thus

$$[\mathbb{P}(V) \circlearrowleft G] \neq [\mathbb{P}(V') \circlearrowleft G] \in \mathrm{Burn}_2(G),$$

and we conclude that the G-actions are not birational.

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